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Some new characterizations of the Bloch space

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Abstract
We obtain some new characterizations for the Bloch space on the open unit disk in the complex plane $\mathbb{C}$ and the open unit ball of $\mathbb{C}^n$.
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1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\mathbb{B}$ the open unit ball of the complex vector space $\mathbb{C}^n$, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$ and $H(\mathbb{B})$ the class of all holomorphic functions on $\mathbb{B}$.

For $f \in C^1(\mathbb{B})$, the invariant gradient $\widetilde{\nabla} f$ is defined by

$$(\widetilde{\nabla} f)(z) = \nabla(f \circ \varphi_z)(0),$$

where

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z)\right)$$

is the complex gradient of f.

A holomorphic function f in $\mathbb{D}$ is said to belong to the Bloch space $\mathcal{B}(\mathbb{D})$ if

$$\|f\|_{\mathcal{B}} = \left|f(0)\right| + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|f'(z)\right| < \infty.$$

Under the above norm, $\mathcal{B}(\mathbb{D})$ is a Banach space (see, *e.g.* [1]). On the unit ball, the Bloch space $\mathcal{B}(\mathbb{B})$, which was introduced by Hahn in [2], is the space of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{B}} \sup_{w\in\mathbb{C}^n\setminus\{0\}} \frac{|\langle\nabla f(z), \overline{w}\rangle|}{\sqrt{\frac{n+1}{2}\frac{(1-|z|^2)|w|^2+|\langle w,z\rangle|^2}{(1-|z|^2)^2}}} < \infty.$$

For some classical results on Bloch spaces see [3] and [4].

It is well known that $f \in \mathcal{B}(\mathbb{B})$ if and only if (see, *e.g.* [5])

$$\sup_{z\in\mathbb{B}}\left(1-|z|^2\right)\left|\nabla f(z)\right| < \infty.$$

For $\alpha > 0$, an $f \in H(\mathbb{B})$ is said to belong to the α-Bloch space, denoted by $\mathcal{B}^\alpha(\mathbb{B})$, if

$$\sup_{z\in\mathbb{B}}\left(1-|z|^2\right)^\alpha\left|\nabla f(z)\right| < \infty.$$

When $\alpha = 1$, the α-Bloch space is the classical Bloch space.

It is of some importance to give new characterizations for a function space, since, for example, it can be useful in the study of operators acting on the space. For example, by using the last expression, it is difficult to study composition operators on α-Bloch space. However, in [6] Zhang and Xu introduced the metric

$$F_z^\alpha(w) = \sqrt{\frac{n+1}{2}}\frac{\sqrt{\lambda_\alpha(|z|)|w|^2 + (1-\lambda_\alpha(|z|))|\langle w,z\rangle|^2/|z|^2}}{(1-|z|^2)^\alpha}$$

and proved that $f \in \mathcal{B}^\alpha(\mathbb{B})$ if and only if

$$\sup_{z,w\in\mathbb{C}^n\setminus\{0\}}\frac{|\nabla f(z)w|}{F_z^\alpha(w)} < \infty, \tag{1}$$

and by using (1), they completely characterized the boundedness and compactness of composition operators on α-Bloch spaces. For some other results on operators on Bloch-type spaces see, for example, [7–21] and the references therein.

For $f \in H(\mathbb{B})$, in [22] was proved that $f \in \mathcal{B}(\mathbb{B})$ if and only if

$$M_1 = \sup_{\substack{z,w\in\mathbb{B}\\ z\neq w}}\left(1-|z|^2\right)^{1/2}\left(1-|w|^2\right)^{1/2}\cdot\frac{|f(z)-f(w)|}{|w-P_wz-s_wQ_wz|} < \infty. \tag{2}$$

Somewhat later, in [23] it was proved that $f \in \mathcal{B}(\mathbb{B})$ if and only if

$$M_2 = \sup_{\substack{z,w\in\mathbb{B}\\ z\neq w}}\left(1-|z|^2\right)^{1/2}\left(1-|w|^2\right)^{1/2}\cdot\frac{|f(z)-f(w)|}{|z-w|} < \infty, \tag{3}$$

while in [24] it was proved that $f \in \mathcal{B}(\mathbb{B})$ if and only if

$$M_3 = \sup_{z,w\in\mathbb{B}}\left(1-|z|^2\right)^{1/2}\left(1-|w|^2\right)^{1/2}\cdot\frac{|f(z)-f(w)|}{|1-\langle z,w\rangle|} < \infty.$$

These characterizations can be seen as derivative-free characterizations of the Bloch space on the unit ball. For the case of the unit polydisk, see [25] and [26]. For more characterizations of Bloch-type spaces in the unit disk, unit polydisk and unit ball, see, for example, [5, 22–24, 27–49].

In this paper, we give some new characterizations for the Bloch space. In Section 2, we give some preliminary results which are used in the proofs of our main results. In Section 3, we give two new characterizations for the Bloch space on $\mathbb{D}$. In Section 4, we give four new characterizations for the Bloch space in the unit ball $\mathbb{B}$, which, among others, generalize the corollaries in Section 3.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the next. We say that two quantities $K_1(x)$ and $K_2(x)$ are comparable, if there are positive constants C_1 and C_2 independent of variable x such that

$$C_1K_1(x) \le K_2(x) \le C_2K_1(x).$$

2 Preliminaries and auxiliary results

Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in the complex vector space $\mathbb{C}^n$ and $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$. Let $Aut(\mathbb{B})$ be the group of all biholomorphic selfmaps of $\mathbb{B}$. It is well known that $Aut(\mathbb{B})$ is generated by the unitary operators on $\mathbb{C}^n$ and the involutions φ_a of the form

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle},$$

where $s_a = (1 - |a|^2)^{1/2}$, P_a is the orthogonal projection into the space spanned by $a \in \mathbb{B}$, *i.e.*,

$$P_a z = \frac{\langle z, a\rangle a}{|a|^2}, \qquad |a|^2 = \langle a, a\rangle, \qquad P_0 z = 0$$

and $Q_a = I - P_a$. See [5, 50] for more properties of $\varphi_a(z)$.

Recall that the weighted Bergman space $A^p_\alpha(\mathbb{B})$, where $0 < p < \infty$ and $\alpha > -1$, consists of those functions $f \in H(\mathbb{B})$ for which

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{B}} \bigl|f(z)\bigr|^p \, dv_\alpha(z) = c_\alpha \int_{\mathbb{B}} \bigl|f(z)\bigr|^p \bigl(1 - |z|^2\bigr)^\alpha \, dv(z) < \infty,$$

where $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$, dv is the normalized Lebesgue measure of $\mathbb{B}$ (*i.e.* $v(\mathbb{B}) = 1$). When $n = 1$, we denote dv_α by dA_α. When $\alpha = 0$, we get the classical Bergman space, which will be denoted by $A^p = A^p(\mathbb{B})$.

Let

$$d\lambda(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}.$$

For any $\psi \in Aut(\mathbb{B})$ and $f \in L^1(\mathbb{B})$,

$$\int_{\mathbb{B}} f(z) \, d\lambda(z) = \int_{\mathbb{B}} f \circ \psi(z) \, d\lambda(z). \tag{4}$$

Thus $d\lambda(z)$ is a Möbius invariant measure (see, *e.g.* [36]).

Next, we quote some well-known results that will be used in the proofs of our main results. We begin with the following characterization of the Bloch space in the unit ball (see [5, 32, 34]).

Lemma 2.1 *Let $0 < p < \infty$. A holomorphic function f is in the Bloch space $\mathcal{B}(\mathbb{B})$ if and only if*

$$\sup_{a \in \mathbb{B}} \bigl\|f \circ \varphi_a - f(a)\bigr\|_{A^p} < \infty.$$

In [51], are proved the following two characterizations for the weighted Bergman space in the unit ball.

Lemma 2.2 *Assume that* $0 < p < \infty$, $\alpha > -1$ *and* $f \in H(\mathbb{B})$. *If* β *and* γ *are real parameters such that*

$$\beta + \gamma = \alpha + p - (n+1) \tag{5}$$

and

$$-1 < \beta < p - (n+1), \qquad -1 < \gamma < p - (n+1), \tag{6}$$

then the following statements are equivalent:

(a) $f \in A^p_\alpha(\mathbb{B})$;

(b)

$$Q_1(f) := \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(z) - f(w)|^p}{|z-w|^p}\, dv_\beta(z)\, dv_\gamma(w) < \infty; \tag{7}$$

(c)

$$Q_2(f) := \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(z) - f(w)|^p}{|1 - \langle z, w\rangle|^p}\, dv_\beta(z)\, dv_\gamma(w) < \infty. \tag{8}$$

Moreover, the quantities $Q_1(f)$, $Q_2(f)$, *and* $\|f\|^p_{A^p_\alpha}$ *are comparable.*

Lemma 2.3 [32] *Assume that* $f \in H(\mathbb{B})$, $0 < p < \infty$, $-1 < q < \infty$, $0 \le s < \infty$, $0 \le t < p + 2n$, *and* $p + s > n$. *Then for* $a \in \mathbb{B}$,

$$\begin{aligned} &\int_{\mathbb{B}} \frac{|f(z) - f(0)|^p}{|z|^t} \left(1 - |z|^2\right)^q \left(1 - \left|\varphi_a(z)\right|^2\right)^s dv(z) \\ &\quad \le C \int_{\mathbb{B}} |\widetilde{\nabla} f|^p \left(1 - |z|^2\right)^q \left(1 - \left|\varphi_a(z)\right|^2\right)^s dv(z). \end{aligned} \tag{9}$$

Lemma 2.4 [32] *Assume that* $f \in H(\mathbb{B})$ *and* $0 < p < \infty$. *Then* $f \in \mathcal{B}(\mathbb{B})$ *if and only if*

$$\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left|\widetilde{\nabla} f(z)\right|^p \left(1 - \left|\varphi_a(z)\right|^2\right)^{n+1} d\lambda(z) < \infty. \tag{10}$$

The following well-known result can be found in [50] or [5].

Lemma 2.5 *Let* $-1 < t < \infty$ *and* $c \in \mathbb{R}$. *Then there is a positive constant* C *such that*

$$\int_{\mathbb{B}} \frac{(1 - |z|^2)^t}{|1 - \langle z, w\rangle|^{n+1+t+c}}\, dv(z) \begin{cases} \le \frac{C}{(1-|w|^2)^c}, & \text{if } c > 0, \\ \le C \log \frac{e}{1-|w|^2}, & \text{if } c = 0, \\ \text{is bounded}, & \text{if } c < 0, \end{cases}$$

for all $w \in \mathbb{B}$.

3 Characterizations of the Bloch space in the unit disk

In this section, we give two characterizations for the Bloch space in the unit disk as follows.

Theorem 3.1 *Assume that* $f \in H(\mathbb{D})$ *and* $0 < p < \infty$. *If* β *and* γ *are real parameters satisfying the following conditions*:

$$\beta + \gamma = p - 2, \qquad -1 < \beta < p - 2, \qquad -1 < \gamma < p - 2, \tag{11}$$

then the following statements are equivalent:

(a) $f \in \mathcal{B}(\mathbb{D})$;

(b)

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|z - w|^p} \frac{(1 - |a|^2)^2(1 - |z|^2)^\beta(1 - |w|^2)^\gamma}{|1 - \overline{a}z|^{\beta-\gamma+2}|1 - \overline{a}w|^{\gamma-\beta+2}} \, dA(z) \, dA(w) < \infty;$$

(c)

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|1 - \overline{z}w|^p} \frac{(1 - |a|^2)^2(1 - |z|^2)^\beta(1 - |w|^2)^\gamma}{|1 - \overline{a}z|^{\beta-\gamma+2}|1 - \overline{a}w|^{\gamma-\beta+2}} \, dA(z) \, dA(w) < \infty.$$

Proof By taking $n = 1$, $\alpha = 0$ in Lemma 2.2, we see that $f \in A^p(\mathbb{D})$ if and only if

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|z - w|^p} \left(1 - |z|^2\right)^\beta \left(1 - |w|^2\right)^\gamma dA(z) \, dA(w) < \infty \tag{12}$$

and if and only if

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|1 - \overline{z}w|^p} \left(1 - |z|^2\right)^\beta \left(1 - |w|^2\right)^\gamma dA(z) \, dA(w) < \infty, \tag{13}$$

when the conditions in (11) hold.

Replacing f by $f \circ \varphi_a - f(a)$ in (12) and (13), and using Lemma 2.1, we conclude that $f \in \mathcal{B}(\mathbb{D})$ if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f \circ \varphi_a(z) - f \circ \varphi_a(w)|^p}{|z - w|^p} \left(1 - |z|^2\right)^\beta \left(1 - |w|^2\right)^\gamma dA(z) \, dA(w) < \infty, \tag{14}$$

which is equivalent to

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f \circ \varphi_a(z) - f \circ \varphi_a(w)|^p}{|1 - \overline{z}w|^p} \left(1 - |z|^2\right)^\beta \left(1 - |w|^2\right)^\gamma dA(z) \, dA(w) < \infty. \tag{15}$$

Using the change of variables $z \mapsto \varphi_a(z)$, $w \mapsto \varphi_a(w)$ and the following equalities (see, *e.g.* [1]):

$$\left|\varphi_a(z) - \varphi_a(w)\right| = \frac{|z - w|(1 - |a|^2)}{|1 - \overline{a}w||1 - \overline{a}z|}$$

and

$$\left|1 - \overline{\varphi_a(z)}\varphi_a(w)\right| = \frac{|1 - \overline{z}w|(1 - |a|^2)}{|1 - \overline{a}w||1 - \overline{a}z|},$$

we see that the double integrals on the left side of (14) and (15) are equivalent to

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p}{|z-w|^p}\frac{(1-|a|^2)^2(1-|z|^2)^\beta(1-|w|^2)^\gamma}{|1-\overline{a}z|^{\beta-\gamma+2}|1-\overline{a}w|^{\gamma-\beta+2}}\,dA(z)\,dA(w)$$

and

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p}{|1-\overline{z}w|^p}\frac{(1-|a|^2)^2(1-|z|^2)^\beta(1-|w|^2)^\gamma}{|1-\overline{a}z|^{\beta-\gamma+2}|1-\overline{a}w|^{\gamma-\beta+2}}\,dA(z)\,dA(w),$$

respectively. Therefore, $f\in\mathcal{B}(\mathbb{D})$ if and only if (b) holds, and if and only if (c) holds, as desired. □

Taking $\beta=\gamma=p/2-1$ in Theorem 3.1, we easily get the following corollary.

Corollary 3.1 *Assume that $f\in H(\mathbb{D})$ and $2<p<\infty$. Then the following statements are equivalent*:

(a) $f\in\mathcal{B}(\mathbb{D})$;

(b)

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\int_{\mathbb{D}}\left(\frac{|f(z)-f(w)|}{|z-w|}\right)^p\left(1-\left|\varphi_a(z)\right|^2\right)\left(1-\left|\varphi_a(w)\right|^2\right)dA_t(z)\,dA_t(w)<\infty;$$

(c)

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\int_{\mathbb{D}}\left(\frac{|f(z)-f(w)|}{|1-\overline{z}w|}\right)^p\left(1-\left|\varphi_a(z)\right|^2\right)\left(1-\left|\varphi_a(w)\right|^2\right)dA_t(z)\,dA_t(w)<\infty,$$

where $t=(p-4)/2$.

Taking $\beta=p/2-2$, $\gamma=p/2$ in Theorem 3.1, we can easily get the following result.

Corollary 3.2 *Assume that $f\in H(\mathbb{D})$ and $4<p<\infty$. Then the following statements are equivalent*:

(a) $f\in\mathcal{B}(\mathbb{D})$;

(b)

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\int_{\mathbb{D}}\left(\frac{|f(z)-f(w)|}{|z-w|}\right)^p\left(1-\left|\varphi_a(w)\right|^2\right)^2dA_t(z)\,dA_t(w)<\infty;$$

(c)

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\int_{\mathbb{D}}\left(\frac{|f(z)-f(w)|}{|1-\overline{z}w|}\right)^p\left(1-\left|\varphi_a(w)\right|^2\right)^2dA_t(z)\,dA_t(w)<\infty,$$

where $t=(p-4)/2$.

4 Characterizations of the Bloch space in the unit ball

In this section, we generalize Corollaries 3.1 and 3.2 in the setting of the unit ball.

Theorem 4.1 *Assume that $f \in H(\mathbb{B})$ and $n+1 < p < \infty$. Then $f \in \mathcal{B}(\mathbb{B})$ if and only if*

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \left(\frac{|f(z) - f(w)|}{|1 - \langle z, w\rangle|} \right)^p \left(1 - \left|\varphi_a(z)\right|^2\right)^{\frac{n+1}{2}} \left(1 - \left|\varphi_a(w)\right|^2\right)^{\frac{n+1}{2}} \, dv_t(z)\, dv_t(w) < \infty,$$

where $t = (p - 2(n+1))/2$.

Proof Let $\alpha = 0$ and $\beta = \gamma = (p - (n+1))/2$ in Lemma 2.2. We see that $f \in A^p_\alpha(\mathbb{B})$ if and only if

$$\int_{\mathbb{B}} \int_{\mathbb{B}} \left(\frac{|f(z) - f(w)|}{|1 - \langle z, w\rangle|} \left(1 - |z|^2\right)^{1/2} \left(1 - |w|^2\right)^{1/2} \right)^p dv_k(z)\, dv_k(w) < \infty,$$

where $k = -(n+1)/2$.

From this and by Lemma 2.1 we see that $f \in \mathcal{B}(\mathbb{B})$ if and only if

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \left(\frac{|f\circ\varphi_a(z) - f\circ\varphi_a(w)|}{|1 - \langle z, w\rangle|} \left(1 - |z|^2\right)^{\frac{1}{2}} \left(1 - |w|^2\right)^{\frac{1}{2}} \right)^p dv_k(z)\, dv_k(w) < \infty.$$

Using the change of variables $z \mapsto \varphi_a(z)$, $w \mapsto \varphi_a(w)$ and (4), we see that the left side of the last inequality is equivalent to

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(z) - f(w)|^p (1 - |\varphi_a(z)|^2)^{\frac{p+n+1}{2}} (1 - |\varphi_a(w)|^2)^{\frac{p+n+1}{2}}}{|1 - \langle \varphi_a(z), \varphi_a(w)\rangle|^p} \, d\lambda(z)\, d\lambda(w). \tag{16}$$

The result follows by using the equalities (see, *e.g.* [5])

$$1 - \left\langle \varphi_a(z), \varphi_a(w)\right\rangle = \frac{(1 - \langle a, a\rangle)(1 - \langle z, w\rangle)}{(1 - \langle z, a\rangle)(1 - \langle a, w\rangle)}$$

and

$$1 - \left|\varphi_a(z)\right|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a\rangle|^2},$$

in (16). □

Theorem 4.2 *Assume that $f \in H(\mathbb{B})$ and $2n < p < \infty$. Then $f \in \mathcal{B}(\mathbb{B})$ if and only if*

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \left(\frac{|f(z) - f(w)|}{|w - P_w z - s_w Q_w z|} \right)^p \left(1 - \left|\varphi_a(z)\right|^2\right)^{\frac{n+1}{2}} \left(1 - \left|\varphi_a(w)\right|^2\right)^{\frac{n+1}{2}} \, dv_t(z)\, dv_t(w) < \infty, \tag{17}$$

where $t = (p - 2(n+1))/2$.

Proof Suppose that (17) holds. Since

$$\frac{1}{|1 - \langle z, w\rangle|} \le \frac{1}{|w - P_w z - s_w Q_w z|}, \quad z, w \in \mathbb{B}. \tag{18}$$

Then by Theorem 4.1 and (18) we see that $f \in \mathcal{B}(\mathbb{B})$.

Conversely, suppose that $f \in \mathcal{B}(\mathbb{B})$. By using the change of variables $z \to \varphi_w(u)$, Lemma 2.3 and the following equality (see [5]):

$$\frac{1}{1-\langle \varphi_w(u), w\rangle} = \frac{1-\langle u,w\rangle}{1-|w|^2}, \quad u,w \in \mathbb{B},$$

we have

$$\begin{aligned}
&\int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|f(z)-f(w)|^p(1-|\varphi_a(z)|^2)^{\frac{n+1}{2}}(1-|\varphi_a(w)|^2)^{\frac{n+1}{2}}}{|w-P_wz-s_wQ_wz|^p}\,dv_t(z)\,dv_t(w) \\
&\quad= \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|f(z)-f(w)|^p(1-|\varphi_a(z)|^2)^{\frac{n+1}{2}}(1-|\varphi_a(w)|^2)^{\frac{n+1}{2}}}{|\varphi_w(z)|^p|1-\langle w,z\rangle|^p}\,dv_t(z)\,dv_t(w) \\
&\quad= \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|f\circ\varphi_w(u)-f\circ\varphi_w(0)|^p}{|u|^p}\bigl(1-\bigl|\varphi_a\bigl(\varphi_w(u)\bigr)\bigr|^2\bigr)^{\frac{n+1}{2}}\,dv_t(u)\bigl(1-\bigl|\varphi_a(w)\bigr|^2\bigr)^{\frac{n+1}{2}}\,d\lambda(w) \\
&\quad\le C\int_{\mathbb{B}}\int_{\mathbb{B}} \bigl|\widetilde{\nabla} f\circ\varphi_w(u)\bigr|^p\bigl(1-\bigl|\varphi_a\bigl(\varphi_w(u)\bigr)\bigr|^2\bigr)^{\frac{n+1}{2}}\,dv_t(u)\bigl(1-\bigl|\varphi_a(w)\bigr|^2\bigr)^{\frac{n+1}{2}}\,d\lambda(w) \\
&\quad\le C\int_{\mathbb{B}}\int_{\mathbb{B}} \bigl|\widetilde{\nabla} f(z)\bigr|^p\bigl(1-\bigl|\varphi_w(z)\bigr|^2\bigr)^{n+1+t}\bigl(1-\bigl|\varphi_a(z)\bigr|^2\bigr)^{\frac{n+1}{2}}\,d\lambda(z)\bigl(1-\bigl|\varphi_a(w)\bigr|^2\bigr)^{\frac{n+1}{2}}\,d\lambda(w) \\
&\quad\le CK\int_{\mathbb{B}} \bigl|\widetilde{\nabla} f(z)\bigr|^p\bigl(1-\bigl|\varphi_a(z)\bigr|^2\bigr)^{n+1}\,d\lambda(z),
\end{aligned} \tag{19}$$

where

$$K = \sup_{a,z\in\mathbb{B}}\int_{\mathbb{B}} \frac{1}{(1-|\varphi_a(z)|^2)^{\frac{n+1}{2}}}\bigl(1-\bigl|\varphi_w(z)\bigr|^2\bigr)^{n+1+t}\bigl(1-\bigl|\varphi_a(w)\bigr|^2\bigr)^{\frac{n+1}{2}}\,d\lambda(w).$$

Employing the change of variables $w \mapsto \varphi_z(u)$ and using the fact that $|\varphi_z(w)| = |\varphi_w(z)|$ we have

$$K = \sup_{a,z\in\mathbb{B}}\int_{\mathbb{B}} \frac{1}{(1-|\varphi_z(a)|^2)^{\frac{n+1}{2}}}\bigl(1-|u|^2\bigr)^{n+1+t}\bigl(1-\bigl|\varphi_a\bigl(\varphi_z(u)\bigr)\bigr|^2\bigr)^{\frac{n+1}{2}}\,d\lambda(u).$$

It follows from Lemma 2.5 and the fact that $|(\varphi_a\circ\varphi_z)(u)| = |\varphi_{\varphi_z(a)}(u)|$ (see [5]) that

$$\begin{aligned}
K &= \sup_{a,z\in\mathbb{B}}\int_{\mathbb{B}} \frac{1}{(1-|\varphi_z(a)|^2)^{\frac{n+1}{2}}}\bigl(1-\bigl|\varphi_{\varphi_z(a)}(u)\bigr|^2\bigr)^{\frac{n+1}{2}}\,dv_t(u) \\
&= \sup_{a,z\in\mathbb{B}}\int_{\mathbb{B}} \frac{(1-|u|^2)^{\frac{n+1}{2}}}{|1-\langle u,\varphi_z(a)\rangle|^{n+1}}\,dv_t(u) = \sup_{w\in\mathbb{B}}\int_{\mathbb{B}} \frac{(1-|u|^2)^{\frac{n+1}{2}}}{|1-\langle u,w\rangle|^{n+1}}\,dv_t(u) < \infty.
\end{aligned} \tag{20}$$

Combining (19) with (20), the result follows from Lemma 2.4. □

By choosing $\alpha = 0$, $\gamma = p/2$, and $\beta = p/2-(n+1)$ in Lemma 2.2, similarly to the proof of Theorem 4.1 is obtained the following result.

Theorem 4.3 *Assume that $f \in H(\mathbb{B})$ and $2(n+1) < p < \infty$. Then $f \in \mathcal{B}(\mathbb{B})$ if and only if*

$$\sup_{a\in\mathbb{B}}\int_{\mathbb{B}}\int_{\mathbb{B}}\left(\frac{|f(z)-f(w)|}{|1-\langle z,w\rangle|}\right)^p\bigl(1-\bigl|\varphi_a(w)\bigr|^2\bigr)^{n+1}\,dv_t(z)\,dv_t(w) < \infty,$$

where $t = (p-2(n+1))/2$.

Using Theorem 4.3, similarly to the proof of Theorem 4.2 is proved the following result.

Theorem 4.4 *Assume that* $f \in H(\mathbb{B})$ *and* $2(n+1) < p < \infty$. *Then* $f \in \mathcal{B}(\mathbb{B})$ *if and only if*

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \left(\frac{|f(z) - f(w)|}{|w - P_w z - s_w Q_w z|} \right)^p \left(1 - \left|\varphi_a(w)\right|^2\right)^{n+1} dv_t(z)\, dv_t(w) < \infty,$$

where $t = (p - 2(n+1))/2$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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